

## QUANTUM COMPUTING WITH SUPERQUBITS

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**Abstract**

We analyze some aspects of quantum computing with super-qubits (squbits). We propose the analogue of a superfield formalism, and give a physical interpretation for the Grassmann coefficients in the squbit expansion as fermionic creation operators of an auxiliary quantum system. In the simplest case the squbit is a superposition of one Bose  $\otimes$  Bose and one Fermi  $\otimes$  Fermi state, and its norm is invariant under a  $U(2)$  group realized with Clifford-valued matrices. This case can be generalized to a superposition of  $n_B$  bosonic and  $n_F$  fermionic states, with a norm invariant under  $U(n_B + n_F)$ . Entanglement between squbits, super quantum gates and teleportation are discussed.

# 1 Introduction

In recent work [1, 2, 3] a measure for tripartite qubit entanglement has been given in terms of Cayley's hyperdeterminant [4], a generalization of the usual determinant of a square matrix to the case of cubic matrices. A supersymmetric generalization of the hyperdeterminant was found in [5] for cubic supermatrices and inspired the construction of super-qubits of ref. [6].

Here we present a somewhat different formulation of super-qubits, renamed *squbits* for short. This formulation allows for a physical interpretation of the expansion coefficients on the super-Hilbert space basis. In order to construct a superposition of bosonic and fermionic states one needs to circumvent the superselection rules by introducing a second (auxiliary) quantum system, containing bosons and fermions. Then we consider the bosonic subspace in the tensor product of the two super-Hilbert spaces, so that squbits have bosonic statistics. The Grassmann coefficients of ref.[6] become here fermion creation operators of the auxiliary system.

The plan of the paper is as follows: in Section 2 we introduce squbits as a superfield expansion, and point out the difficulty for a physical interpretation (and realization). An auxiliary bose-fermi quantum system is then introduced, and Clifford squbits are defined. Tensor products of these squbits are examined in Section 3. Mixed states and their tensor products are the subject of Section 4, where the properties of the corresponding density matrix  $\rho$  are studied. As for usual qubits,  $\rho^2 = \rho$  is the condition for purity. Entanglement of squbits is treated in a systematic way in Section 5. Unitary 1-squbit and 2-squbit supergates are introduced in Section 6, including the supersymmetry gate that exchanges bosonic and fermionic states. In Section 7 we discuss super-teleportation, and observe that some caution is necessary in using correlations in the auxiliary quantum system between squbits that are separated in space. Finally Section 8 contains conclusions and outlook.

## 2 Squbits

### 2.1 Grassmann squbits

As in ref. [6], we consider superpositions of bosonic and fermionic states, with coefficients being Grassmann numbers rather than complex numbers. We can choose from the start the orthonormal basis of the super-Hilbert space to contain an *equal number* of bosonic and fermionic states, preparing the ground for supersymmetry. In fact, using an expansion in the anticommuting Grassmann coordinates  $\theta_i$ , leads to a superfield expansion of the squbit:

$$|\psi\rangle = b|B\rangle + f_i\theta_i|F_i\rangle + b_{ij}\theta_i\theta_j|B_{ij}\rangle + f_{ijk}\theta_i\theta_j\theta_k|F_{ijk}\rangle + \dots \quad (2.1)$$

(sum on repeated indices, with  $i < j < k \dots$ ) where the  $|B\rangle$  states are bosonic and the  $|F\rangle$  states are fermionic. All their coefficients  $b$  and  $f$  are complex. For  $N$  Grassmann coordinates  $\theta_i$ , there are  $2^{N-1}$  bosonic and  $2^{N-1}$  fermionic states in

the above expansion. Note that the denomination “squbit” is not really justified in this case, since  $|\psi\rangle$  does not reduce to the usual qubit when all  $\theta$ ’s are set to zero. In forthcoming Sections we will consider more general squbits that indeed reduce to the usual qubits or qudits (states of a  $d$ -level quantum system). All of these will be called squbits, even if their bosonic part is a qudit.

The adjoint of  $|\psi\rangle$  is:

$$\langle\psi| = \langle B|b^* + \langle F_i|\bar{\theta}_i f_i^* + \langle B_{ij}|\bar{\theta}_j \bar{\theta}_i b_{ij}^* + \langle F_{ijk}|\bar{\theta}_k \bar{\theta}_j \bar{\theta}_i f_{ijk}^* + \dots \quad (2.2)$$

where we have used the usual properties of conjugation of fermionic quantities (i.e. we do not use the superstar conjugation of ref. [6]).

The usual norm of  $|\psi\rangle$  is then a Grassmann number:

$$\langle\psi|\psi\rangle = |b|^2 + |f_i|^2 \bar{\theta}_i \theta_i + |b_{ij}|^2 \bar{\theta}_j \bar{\theta}_i \theta_i \theta_j + |f_{ijk}|^2 \bar{\theta}_k \bar{\theta}_j \bar{\theta}_i \theta_i \theta_j \theta_k + \dots \quad (2.3)$$

By integration on the Grassmann coordinates with an appropriate weight, we can define a *positive* super-norm as :

$$\langle\langle\psi|\psi\rangle\rangle \equiv \int e^{\sum_i \bar{\theta}_i \theta_i} \langle\psi|\psi\rangle \Pi_i d\bar{\theta}_i d\theta_i = |b|^2 + \sum |f_i|^2 + \sum_{i<j} |b_{ij}|^2 + \sum_{i<j<k} |f_{ijk}|^2 + \dots \quad (2.4)$$

admitting the interpretation as a sum of probabilities. Namely, the probability to “find the system” in one of the basis states is just the square modulus of the corresponding complex coefficient, the state being normalized as  $\langle\langle\psi|\psi\rangle\rangle = 1$ . For a discussion on norms on super Hilbert spaces see for example ref. [7].

Note that, even if we are able to give a probabilistic interpretation of the coefficients in the squbit expansion, a physical realization of the squbit remains problematic, see also ref. [6]. The auxiliary anticommuting coordinates  $\theta_i$  are introduced in order that all terms in the superposition in (2.1) be bosonic, so that the superqubit itself has Bose statistics. This is necessary to circumvent superselection rules (see for ex. [8]), which forbid a superposition of bose states with fermi states.

To realize the superposition in (2.1) as a quantum mechanical construction, we need a further quantum mechanical system (with bosons and fermions) that we can tensor with the original ( $|B\rangle, |F\rangle$ ) system. Then we can restrict our attention to the bosonic subsector of this tensor product.

In the next subsection we use dynamical  $\theta_i$ ’s that are part of a Clifford algebra. This provides, at least in principle, a way to a physical realization of squbits.

## 2.2 Clifford squbits

We assume that we have a quantum mechanical system whose Hilbert states are either fermionic or bosonic. For example, we can consider quantum electrodynamics with photons and electrons. We denote by  $|B_I\rangle$  the states for a single photon (we can assume that  $|B_I\rangle$  are the helicity states) and we denote by  $|F_A\rangle$  the states for a single electron. They are produced by acting with the field operators

$A_I, \psi_A$  on the vacuum of the Fock space. The Hilbert space  $\mathcal{H}_{BF}$  of physical states decomposes into bosonic and fermionic subspaces:  $\mathcal{H}_{BF} = \mathcal{H}_B \oplus \mathcal{H}_F$ . We would like to combine those states in order to have superpositions of bosons and fermions. For this we consider an auxiliary quantum mechanical system described by the action  $S = \int dt \bar{\theta}^i \dot{\theta}_i$ . This system has only zero mode states. The bosonic states are  $|0\rangle, \theta_i \theta_j |0\rangle, \dots$  and the fermionic states are  $\theta_i |0\rangle, \theta_i \theta_j \theta_k |0\rangle, \dots$ , where  $|0\rangle$  is the Fock vacuum of the system. The operators  $\theta_i, \bar{\theta}^i$  satisfy the Clifford algebra  $\{\bar{\theta}^i, \theta_j\} = \delta_j^i$ ,  $\{\theta_i, \theta_j\} = 0$ ,  $\{\bar{\theta}^i, \bar{\theta}^j\} = 0$ . The Hilbert space  $\mathcal{H}^\theta$  for this system is decomposed into two parts  $\mathcal{H}_B^\theta \oplus \mathcal{H}_F^\theta$  which again cannot be mixed because of the superselection rules. Moreover, to tensor states of the  $\mathcal{H}^\theta$  system we follow the rule:

$$\theta_i \theta_j \dots |0\rangle \otimes \theta_k \theta_l \dots |0\rangle = \theta_i \theta_j \dots \theta_k \theta_l \dots |0\rangle \quad (2.5)$$

This rule is justified in the context of quantum field theory, where for example the tensor product of two one-particle states is a two particle state and so on.

Considering now the combined system with Hilbert space  $\mathcal{H}_{BF} \otimes \mathcal{H}^\theta$  we can construct superpositions involving bosons and fermions of the original ( $|B\rangle, |F\rangle$ ) system. Supposing that there are  $2^{N-1}$  bosons  $|B_I\rangle$  and  $2^{N-1}$  fermions  $|F_A\rangle$ , we can organize the states according to the binomial expansion

$$|\psi\rangle = b|0\rangle \otimes |B\rangle + f_i \theta_i |0\rangle \otimes |F_i\rangle + b_{ij} \theta_i \theta_j |0\rangle \otimes |B_{[ij]}\rangle + f_{ijk} \theta_i \theta_j \theta_k |0\rangle \otimes |F_{[ijk]}\rangle + \dots \quad (2.6)$$

where we have rearranged the bosonic states  $|B_I\rangle$  into the multiplet of bosonic states ( $|B\rangle, |B_i\rangle, |B_{[ij]}\rangle, \dots$ ) and the fermionic states into an analogous multiplet. In this way all the terms of the expansion have the same (bosonic) statistics, and we are considering only those vectors belonging to the bosonic subspace of  $\mathcal{H}_{BF} \otimes \mathcal{H}^\theta$ , i.e. to the Hilbert space

$$\mathcal{H} = \left( \mathcal{H}_B \otimes \mathcal{H}_B^\theta \right) \oplus \left( \mathcal{H}_F \otimes \mathcal{H}_F^\theta \right)$$

This is a *bona fide* Hilbert space with norm induced by the norms in  $\mathcal{H}_{BF}$  and  $\mathcal{H}^\theta$ :

$$\langle \psi | \psi \rangle = |b|^2 + \sum_i |f_i|^2 + \sum_{i < j} |b_{ij}|^2 + \sum_{i < j < k} |f_{ijk}|^2 + \dots \quad (2.7)$$

as can be checked by using

$$\langle \psi | = \langle B | \otimes \langle 0 | b^* + \langle F_i | \otimes \langle 0 | \bar{\theta}_i f_i^* + \langle B_{ij} | \otimes \langle 0 | \bar{\theta}_j \bar{\theta}_i b_{ij}^* + \langle F_{ijk} | \otimes \langle 0 | \bar{\theta}_k \bar{\theta}_j \bar{\theta}_i f_{ijk}^* + \dots \quad (2.8)$$

and the Clifford anticommutation rules.

Thus the auxiliary  $\theta$  system allows the dynamical mixing between fermions and bosons of the  $\mathcal{H}_{BF}$  system, with a well-defined positive norm, necessary for a probabilistic interpretation.

The simplest squibit is constructed using just two states  $|B\rangle$  and  $|F\rangle$  and introducing a single  $\theta$ . The generic superstate is then

$$|\psi\rangle = b|0\rangle \otimes |B\rangle + f\theta|0\rangle \otimes |F\rangle \quad (2.9)$$

where  $b, f$  are the complex amplitudes. The positive-definite norm of the state is  $|b|^2 + |f|^2$ . The Clifford coordinate  $\theta$ , in contrast with the usual superspace coordinate  $\theta$ , is a dynamical quantity and we can act upon it with quantum operations.

### 3 Tensor Products

Let us consider the tensor product of the simplest squibit with itself. Imagining to realize squibits with identical particles, we consider the symmetrized tensor product  $\frac{1}{2}(|\psi\rangle \otimes |\psi'\rangle + |\psi'\rangle \otimes |\psi\rangle)$  (since  $|\psi\rangle$  is a bosonic state):

$$\begin{aligned} [|\psi\rangle \otimes |\psi'\rangle]_{symm} &= [(b|0\rangle \otimes |B\rangle + f\theta|0\rangle \otimes |F\rangle) \otimes (b'|0\rangle \otimes |B\rangle + f'\theta|0\rangle \otimes |F\rangle)]_{symm} = \\ &= bb'|00\rangle \otimes |BB\rangle + (bf' + b'f)\theta|00\rangle \otimes |BF\rangle, \end{aligned} \quad (3.1)$$

where  $|00\rangle \equiv |0\rangle \otimes |0\rangle$ ,  $|BB\rangle \equiv |B\rangle \otimes |B\rangle$  and  $|BF\rangle \equiv \frac{1}{2}(|B\rangle \otimes |F\rangle + |F\rangle \otimes |B\rangle)$ . Moreover  $\theta|0\rangle \otimes \theta|0\rangle = 0$  because of the rule (2.5) and  $\theta^2 = 0$ .

Identifying the vacuum of the Fock space  $|0\rangle$  with the tensor product of the vacuum of the single particle Hilbert spaces  $|00\rangle$ , we can rewrite eq. (3.1) as follows

$$[|\psi\rangle \otimes |\psi'\rangle]_{symm} = c|0\rangle \otimes |BB\rangle + d\theta|0\rangle \otimes |BF\rangle. \quad (3.2)$$

where  $c = bb'/\sqrt{|b|^2|b'|^2 + |bf' + b'f|^2}$  and  $d = (bf' + b'f)/\sqrt{|b|^2|b'|^2 + |bf' + b'f|^2}$ . The tensor product has produced again a bosonic state since we have tensored the bosonic subspaces  $(\mathcal{H}_B \otimes \mathcal{H}_B^\theta) \oplus (\mathcal{H}_F \otimes \mathcal{H}_F^\theta)$  leading to the Hilbert space

$$\left(\mathcal{H}_B \otimes \mathcal{H}_B^\theta \otimes \mathcal{H}_B \otimes \mathcal{H}_B^\theta\right) \oplus \left(\mathcal{H}_B \otimes \mathcal{H}_B^\theta \otimes \mathcal{H}_F \otimes \mathcal{H}_F^\theta\right).$$

The tensor product of  $\mathcal{H}_F \otimes \mathcal{H}_F^\theta$  with itself does not contain any state since it would involve identical  $\theta$  fermions in the same physical states. The absence of fermi-fermi parts in the tensor product has important consequences, as discussed in the forthcoming sections.

Symmetrized tensor products of copies of the simplest squibit remain in Hilbert subspaces of dimension 2, as we see in the example (3.2). Thus the simplest squibit is not so interesting as element of a super quantum computer realized with identical particles, since in this case any number of these squibits leads always to a 2-state quantum system. However the situation changes if we tensor squibits which are not realised with identical particles: then the tensor product of two simplest squibits is (in the following we omit the  $\theta$ -vacuum  $|0\rangle$ )

$$|\psi\rangle \otimes |\psi'\rangle = bb'|BB\rangle + bf'\theta|BF\rangle + b'f\theta|FB\rangle \quad (3.3)$$

where now  $|BF\rangle = |B\rangle \otimes |F\rangle$  and  $|FB\rangle = |F\rangle \otimes |B\rangle$  are not symmetrized states. This tensor product lives in a Hilbert space with one bosonic and two fermionic base states.

We can consider a more general squibit of the form:

$$|\psi\rangle = \sum_i^n b_i |B_i\rangle + \theta \sum_i^n f_i |F_i\rangle, \quad (3.4)$$

with  $n$  bosonic states  $|B_i\rangle$  and  $n$  fermionic states  $|F_i\rangle$ , combined via a single Clifford coordinate  $\theta$ . Its norm is  $\sum_i^n (|b_i|^2 + |f_i|^2)$ . For identical particles the symmetrized tensor product reads

$$(|\psi\rangle \otimes |\psi'\rangle)_{symm} = \sum_{ij} b_i b'_j |B_i B_j\rangle + \theta \sum_{ij} (b_i f'_j + b'_i f_j) |B_i F_j\rangle \quad (3.5)$$

with  $n(n+1)/2 + n^2$  states surviving in the tensor product. For example when  $n = 2$ , 7 states enter the tensor product.

The  $n = 2$  case is the simplest generalization of the qubit, and indeed its bosonic part is the usual 2-state qubit. Thus we will call it the **N=1 squbit**, since only one  $\theta$  enters its construction, and it reduces to the usual qubit when  $f_i = 0$ . Introducing more  $\theta$ 's leads to higher  $N$  squbits. For example we can consider the  $N = 2$  squbit, with two auxiliary coordinates  $\theta_i$ :

$$|\psi\rangle = \sum_i^2 b_i |B_i\rangle + \sum_i^2 \theta_i f_i |F_i\rangle + \theta_1 \theta_2 |\tilde{B}\rangle, \quad (3.6)$$

i.e. a superqubit with 3 bosonic states  $|B_i\rangle$  and  $|\tilde{B}\rangle$  and 2 fermionic states  $|F_i\rangle$ . Tensor products can be obtained as above by symmetrising  $|\psi\rangle \otimes |\psi'\rangle$  in the case of identical particles.

## 4 Density matrix

The density matrix for a pure state elementary squbit reads:

$$\rho = |\psi\rangle\langle\psi| = |b|^2 |B\rangle\langle B| + b f^* |B\rangle\langle F|\bar{\theta} + f b^* \theta |F\rangle\langle B| + |f|^2 \theta |F\rangle\langle F|\bar{\theta} \quad (4.1)$$

This is a hermitian matrix with  $Tr\rho = |b|^2 + |f|^2 = 1$ .

Mixed states are linear combinations of squbits  $|\psi_\alpha\rangle$ :

$$|\psi_{mixed}\rangle = \sum_\alpha p_\alpha |\psi_\alpha\rangle \quad (4.2)$$

where the classical probabilities  $p_\alpha$  satisfy  $\sum_\alpha p_\alpha = 1$ , and the squbits are normalized as  $\langle\psi_\alpha|\psi_\alpha\rangle = 1$ . The corresponding density operator

$$\rho_{mixed} = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \quad (4.3)$$

has unit trace, and takes the form

$$\rho = a |B\rangle\langle B| + b |B\rangle\langle F|\theta^\dagger + \bar{b} \theta |F\rangle\langle B| + c \theta |F\rangle\langle F|\theta^\dagger \quad (4.4)$$

with  $a, c \in \mathbb{R}$ . The trace normalization  $\text{Tr}\rho = 1$  implies  $c = (1 - a)$ .

The condition that  $\rho$  represents a pure state can be expressed as  $\rho^2 = \rho$ . If we compute the square of the density matrix:

$$\rho^2 = (a^2 + |b|^2)|B\rangle\langle B| + b|B\rangle\langle F|\theta^\dagger + \bar{b}\theta|F\rangle\langle B| + (|b|^2 + (1 - a)^2)\theta|F\rangle\langle F|\theta^\dagger. \quad (4.5)$$

we see that the purity condition amounts to  $a = a^2 + |b|^2$ . In this case  $\rho$  has eigenvalues  $\lambda = 1$  and  $\lambda = 0$  corresponding to the orthogonal eigenvectors:

$$|\lambda = 1\rangle = b|B\rangle + (1 - a)\theta|F\rangle, \quad |\lambda = 0\rangle = b|B\rangle - a\theta|F\rangle \quad (4.6)$$

Let us now consider the tensor product of two mixed states. By using the properties of the anticommuting  $\theta$ 's we obtain

$$\begin{aligned} \rho \otimes \rho' &= aa'|BB\rangle\langle BB| + |BB\rangle(ab'\langle BF| + ba'\langle FB|)\theta^\dagger \\ &\quad + \theta(a\bar{b}'|BF\rangle + \bar{b}a'|FB\rangle)\langle BB| \\ &\quad + \theta[a(1 - a')|BF\rangle\langle BF| + a'(1 - a)|FB\rangle\langle FB| \\ &\quad + b\bar{b}'|BF\rangle\langle FB| + \bar{b}b'|FB\rangle\langle BF|]\theta^\dagger \end{aligned} \quad (4.7)$$

which is not normalized since  $|FF\rangle$  states are absent in the tensor product (because of  $\theta^2 = 0$ ). Then the tensored density matrix must be normalized. After doing so, we can verify that if  $\rho$  and  $\rho'$  describe pure states, then also  $\rho \otimes \rho'$  is a pure state, i.e. if  $\rho^2 = \rho$  and  $\rho'^2 = \rho'$ , we find  $(\rho \otimes \rho')^2 = (\rho \otimes \rho')$ .

For identical particles we must consider the symmetrized tensor product

$$\begin{aligned} [\rho \otimes \rho']_{\text{symm}} &= aa'|BB\rangle\langle BB| + (ab' + a'b)|BB\rangle\langle BF|\theta^\dagger \\ &\quad + (a'\bar{b} + a\bar{b}')\theta|BF\rangle\langle BB| + \left((1 - a)a' + (1 - a')a + b\bar{b}' + b'\bar{b}\right)\theta|BF\rangle\langle BF|\theta^\dagger. \end{aligned} \quad (4.8)$$

with  $|BF\rangle \equiv \frac{1}{2}(|B\rangle \otimes |F\rangle + |F\rangle \otimes |B\rangle)$ .

When  $\rho$  and  $\rho'$  describe pure states, the symmetrized tensor product (even if we normalize it) is not a pure state, i.e.  $(\rho \otimes \rho')_{\text{symm}}^2 \neq (\rho \otimes \rho')_{\text{symm}}$ . Indeed the symmetrization destroys the purity of the combined system, unless  $\rho = \rho'$ . In fact, as discussed in next Section, symmetrization of tensor products induces entanglement, a well known fact since the symmetrized state  $|\psi\rangle \otimes |\psi'\rangle + |\psi'\rangle \otimes |\psi\rangle$  is in general entangled unless  $|\psi\rangle = |\psi'\rangle$ .

## 5 Entanglement

We denote by  $(n_B, n_F, N = 1)$  a squibit of the form

$$|\psi\rangle = \sum_i^{n_B} b_i|B_i\rangle + \theta \sum_I^{n_F} f_I|F_I\rangle, \quad (5.1)$$

and study the tensor product

$$|\psi\rangle \otimes |\psi'\rangle = b_i b'_j |B_i B_j\rangle + b_i f'_j |B_i F_j\rangle + f_i b'_j |F_i B_j\rangle \quad (5.2)$$

A generic state  $|\tilde{\Psi}\rangle$  living in the tensor product of the  $(n_B + n_F)$ -dimensional Hilbert spaces spanned by the basis vectors  $|B_i\rangle, |F_I\rangle$  can be written as

$$|\tilde{\Psi}\rangle = \tilde{b}_{ij} |B_i B_j\rangle + \tilde{f}_{iJ} |B_i F_J\rangle + \tilde{g}_{Ij} |F_I B_j\rangle \quad (5.3)$$

When can we write this state in the form of a tensor product as in (5.2)? Or in other words, when do the equations

$$b_i b'_j = \tilde{b}_{ij}, \quad b_i f'_j = \tilde{f}_{iJ}, \quad f_i b'_j = \tilde{g}_{Ij} \quad (5.4)$$

admit a solution for  $b, f, b', f'$  in terms of  $\tilde{b}, \tilde{f}, \tilde{g}$ ?

First we resort to a simple counting argument. Each  $(n_B, n_F, N = 1)$  squibit depends on  $2n_B + 2n_F - 2$  real numbers (indeed one has to subtract from  $2n_B + 2n_F$  real parameters one arbitrary overall phase and the normalization condition  $\langle\psi|\psi\rangle = 1$ ). Thus the tensor product in (5.2) depends on  $2(2n_B + 2n_F - 2)$  real numbers.

On the other hand the generic state (5.3) depends on  $2n_B^2 + 4n_B n_F - 2$  real parameters. Thus when

$$2(2n_B + 2n_F - 2) \geq 2n_B^2 + 4n_B n_F - 2 \quad (5.5)$$

the number of parameters describing the states  $|\psi\rangle$  and  $|\psi'\rangle$  is sufficient to solve the equations (5.4). This happens only for  $n_B = 0$  (a trivial case, yielding vanishing tensor products) and  $n_B = 1$ . For example, with  $n_B = 1, n_F = 1$ , the generic two-squibit state

$$|\tilde{\Psi}\rangle = \tilde{b}|BB\rangle + \tilde{f}|BF\rangle + \tilde{g}|FB\rangle \quad (5.6)$$

can be written as the product

$$(|B\rangle + \theta \frac{\tilde{g}}{\tilde{b}} |F\rangle) \otimes (\tilde{b}|B\rangle + \theta \tilde{f}|F\rangle) \quad (5.7)$$

(for arbitrary  $n_F$  the same factorization holds with  $\tilde{g}|F\rangle \rightarrow \sum_{I=1}^{n_F} \tilde{g}_I |F_I\rangle$  and  $\tilde{f}|F\rangle \rightarrow \sum_{I=1}^{n_F} \tilde{f}_I |F_I\rangle$ ). Thus no entanglement is possible with tensor products of  $N = 1, n_B = 1$  squibits.

The situation is different for squibits with bigger  $N$  or  $n_B$ . For example, the squibit

$$|\psi\rangle = a |BB\rangle + b \theta_1 \theta_2 |F_1 F_2\rangle. \quad (5.8)$$

cannot be written in a factorized form. In general, the entanglement can be evaluated by computing the partial density matrices and finding their eigenvalues. For  $n_B \geq 2$ , to require that a state living in the tensor product of Hilbert spaces be factorizable amounts to require extra conditions on the parameters  $\tilde{b}, \tilde{f}, \tilde{g}$  (cf. the purely bosonic case, where the equations  $b_i b'_j = \tilde{b}_{ij}$  can be solved only if  $\text{rank}(\tilde{b}_{ij}) = 1$ ).



The first squibit (in order of complication) that may produce entangled states by tensoring is the  $n_B = 2$ ,  $n_F = 1$  squibit already considered in [6].

We can repeat the above analysis for the case of identical particles. We have to symmetrize the state (5.2), but by doing so we obtain a state that is always entangled, unless  $|\psi\rangle = |\psi'\rangle$ . Therefore we consider the tensor product of  $|\psi\rangle$  with itself:

$$|\psi\rangle \otimes |\psi\rangle = b_i b_j |B_i B_j\rangle + 2b_i f_j |B_i F_j\rangle \quad (5.9)$$

where now  $|B_i B_j\rangle \equiv \frac{1}{2}(|B_i\rangle \otimes |B_j\rangle + |B_j\rangle \otimes |B_i\rangle)$  and  $|B_i F_j\rangle \equiv \frac{1}{2}(|B_i\rangle \otimes |F_j\rangle + |F_j\rangle \otimes |B_i\rangle)$ . This tensor product is specified by  $2n_B + 2n_F - 2$  real parameters.

On the other hand we can write the generic state living in the tensor product for identical particles:

$$|\tilde{\Psi}\rangle = \tilde{b}_{ij} |B_i B_j\rangle + \tilde{f}_{ij} |B_i F_j\rangle \quad (5.10)$$

depending on  $n_B(n_B + 1) + 2n_B n_F - 2$  real parameters. When the inequality

$$2n_B + 2n_F - 2 \geq n_B(n_B + 1) + 2n_B n_F - 2 \quad (5.11)$$

holds, the number of parameters in (5.9) is sufficient to solve the equations:

$$b_i b_j = \tilde{b}_{ij}, \quad 2b_i f_j = \tilde{f}_{ij} \quad (5.12)$$

Again for  $n_B = 0$  and  $n_B = 1$  the inequality is satisfied, and the generic state in the symmetric tensor space is factorizable. For example the two-squibit state with  $n_B = 1$ ,  $n_F = 1$ :

$$|\tilde{\Psi}\rangle = \tilde{b} |BB\rangle + \tilde{f} |BF\rangle \quad (5.13)$$

can be factorized as

$$(\tilde{b}^{\frac{1}{2}} |B\rangle + \theta \frac{\tilde{f}}{2\tilde{b}^{\frac{1}{2}}} |F\rangle) \otimes (\tilde{b}^{\frac{1}{2}} |B\rangle + \theta \frac{\tilde{f}}{2\tilde{b}^{\frac{1}{2}}} |F\rangle) \quad (5.14)$$

(for arbitrary  $n_F$  the same factorization holds with  $\tilde{f}|F\rangle \rightarrow \sum_{I=1}^{n_F} \tilde{f}_I |F_I\rangle$ ). Thus we find that no entanglement is possible using multiple squibit states for identical particles. Here too the situation changes when  $n_B \geq 2$  or when  $N > 1$ .

## 6 Supergates

We examine here the structure of linear operations on the  $N = 1$  squibits in (3.4) that preserve their bosonic overall character and their norm, i.e. that send physical states into physical states.

The most general linear operator  $U$  preserving the bosonic character of the  $N = 1$  squibit has the form:

$$U = x_{ij} |B_i\rangle \langle B_j| + y_{ij} \theta |B_i\rangle \langle B_j| \bar{\theta} + r_{ij} \theta |B_i\rangle \langle F_j| + s_{ij} |B_i\rangle \langle F_j| \bar{\theta} + u_{ij} \theta |F_i\rangle \langle B_j| + v_{ij} |F_i\rangle \langle B_j| \bar{\theta} + w_{ij} |F_i\rangle \langle F_j| + z_{ij} \theta |F_i\rangle \langle F_j| \bar{\theta} \quad (6.1)$$

with  $i, j = 1, \dots, n$ . However four terms yield a null result on the squbit, and therefore we can limit ourselves to consider operators of the form:

$$U = x_{ij}|B_i\rangle\langle B_j| + s_{ij}|B_i\rangle\langle F_j|\bar{\theta} + u_{ij}\theta|F_i\rangle\langle B_j| + z_{ij}\theta|F_i\rangle\langle F_j|\bar{\theta} \quad (6.2)$$

It can be checked that the product of two such operators reproduces an operator of the same form. In order to preserve the norm of the squbit this operator must be unitary. Imposing  $U^\dagger U = I$  is equivalent to require the unitarity of the  $2n \times 2n$  matrix:

$$M = \begin{pmatrix} x & s \\ u & z \end{pmatrix} \quad (6.3)$$

Thus the supergate operator, not surprisingly, depends on the entries of the  $U(2n)$  unitary matrix  $M$ , i.e.  $U = U(M)$ , and the map  $M \rightarrow U(M)$  preserves the product:  $U(M)U(M') = U(MM')$ . In other words, the supergate operators  $U$  are representations (on the space of squbits) of the group  $U(2n)$ .

We give now two examples of 1-squbit supergate operators. For simplicity we consider  $N = 1, n_B = 1, n_F = 1$  (the simplest squbit), but the supergates can be immediately generalized to higher values of  $n_B, n_F$ .

### Super-Hadamard gate

$$U_H = \frac{1}{\sqrt{2}} \left( |B\rangle\langle B| + |B\rangle\langle F|\bar{\theta} + \theta|F\rangle\langle B| - \theta|F\rangle\langle F|\bar{\theta} \right). \quad (6.4)$$

### Supersymmetry gate

It is in fact an  $X$  gate for the  $|B\rangle, |F\rangle$  system:

$$U_Q = |B\rangle\langle F|\bar{\theta} + \theta|F\rangle\langle B| \quad (6.5)$$

acting on the basis states as:

$$U_Q|B\rangle = \theta|F\rangle, \quad U_Q\theta|F\rangle = |B\rangle \quad (\text{but } U_Q|F\rangle = 0) \quad (6.6)$$

Note that  $U_Q^2 = I$ . A squbit  $|\psi\rangle = b|B\rangle + f\theta|F\rangle$  is *supersymmetric* if

$$U_Q|\psi\rangle = |\psi\rangle \quad (6.7)$$

i.e. when  $b = f$ .

Note that a supergate  $U$  maps supersymmetric squbits into supersymmetric squbits if and only if it commutes with the supersymmetry gate. In this case  $U$  has the form:

$$U = \rho e^{i\alpha} (|B\rangle\langle B| + |F\rangle\langle F|) + i\sqrt{1 - \rho^2} e^{i\alpha} (|B\rangle\langle F|\bar{\theta} + \theta|F\rangle\langle B|) \quad (6.8)$$

with  $\rho, \alpha \in R$ . It is tempting to speculate that supersymmetry invariance could protect a supersymmetric state against decoherence (but then we would need a “supersymmetric environment”).

## 6.1 CNOT gate

Controlled quantum gates are important examples of multiple qubit gates. Here we consider the simplest example of controlled supergate, acting on two elementary squibits of the form  $|\psi\rangle = b|B\rangle + f\theta|F\rangle$ :

$$U_{CNOT} = \theta|BF\rangle\langle BB| + |BB\rangle\langle BF|\bar{\theta} + \theta|FB\rangle\langle FB|\bar{\theta}, \quad (6.9)$$

a hermitian and unitary operator, hence  $U_{CNOT}^2 = I$ . Recall that the tensor product space in this case has dimension 3, with basis  $|BB\rangle, |\theta|BF\rangle, \theta|FB\rangle$ . There is no  $|FF\rangle$  basis element since  $\theta|F\rangle \otimes \theta|F\rangle = 0$ . Therefore the controlled supergate (6.9) can be represented by a  $3 \times 3$  unitary and hermitian matrix, one dimension less than the usual bosonic CNOT gate. If the first squibit (the control squibit) is in the state  $\theta|F\rangle$ , the  $|B\rangle$  part of the second squibit (the target squibit) remains unchanged, while the information of its  $|F\rangle$  part vanishes because of  $\theta|F\rangle \otimes \theta|F\rangle = 0$ . If the first squibit is in the state  $|B\rangle$ , the supersymmetry operator acts on the second squibit, exchanging  $|B\rangle$  with  $\theta|F\rangle$ . Note that in this gate the control squibit collapses on its bosonic part.

Increasing  $n_B$  and  $n_F$  one can consider more useful controlled supergates, where the control squibit remains unchanged, and 1-squibit supergates are applied to the target squibit.

## 7 Super-teleportation

In this example of teleportation we will use  $N = 2$  squibits of the form

$$|\psi\rangle = b_0|B_0\rangle + b_1|B_1\rangle + \theta_0 f_0|F_0\rangle + \theta_1 f_1|F_1\rangle \quad (7.1)$$

Suppose that the squibit to be teleported between Alice and Bob is a bosonic qubit  $|\psi_C\rangle = a_0|B_0\rangle + a_1|B_1\rangle$ . Moreover, suppose that Alice and Bob have each one element of the super-EPR pair:

$$\theta_0\theta_1(|F_0F_1\rangle - |F_1F_0\rangle)/2$$

obtained from the singlet in the tensor product of two superqubits  $|\psi\rangle$ . Simple algebra yields:

$$\begin{aligned} & \left( a_0|B_0\rangle + a_1|B_1\rangle \right) \otimes \theta_0\theta_1(|F_0F_1\rangle - |F_1F_0\rangle) = \\ & = \theta_0\theta_1 \left( a_0|B_0F_0\rangle \otimes |F_1\rangle - a_0|B_0F_1\rangle \otimes |F_0\rangle + a_1|B_1F_0\rangle \otimes |F_1\rangle - a_1|B_1F_1\rangle \otimes |F_0\rangle \right) = \\ & = \left[ \frac{a_0}{a_1} \theta_0|B_0F_0\rangle + \theta_1|B_0F_1\rangle + \theta_0|B_1F_0\rangle + \frac{a_1}{a_0} \theta_1|B_1F_1\rangle \right] \otimes \left( a_0\theta_0|F_0\rangle + a_1\theta_1|F_1\rangle \right). \end{aligned} \quad (7.2)$$

The second bracket is the squibit of Bob: it is the susy transformed of the original state  $|\psi_C\rangle$ , i.e. the state  $U_Q|\psi_C\rangle$ , where  $U_Q$  is now the supergate that exchanges  $|B_i\rangle$  with  $\theta_i|F_i\rangle$ .

Then Bob applies  $U_Q$  to this state, and recovers the original qubit  $|\psi_C\rangle$ , since  $U_Q^2 = I$ , *without need of any classical communication from Alice*. This clearly violates causality, and is due to our assumption (2.5) that implies  $\theta_0|F_0\rangle \otimes \theta_0|F_0\rangle = 0$  etc. This assumption can make sense only if the two factors in the tensor product originate from the same region in space, as for example in a decay process. Here however we are assuming that the same holds for squbits that are separated by a long distance: in this case the auxiliary quantum system we have been using is inadequate to model squbit correlations, and indeed one should add the  $x$ -dependent modes of the dynamical  $\theta$ 's. Then  $\theta_0(x)|F_0\rangle \otimes \theta_0(x')|F_0\rangle$  does not vanish, and classical communication becomes necessary for teleportation.

## 8 Conclusions and outlook

In this Letter we provide a physical interpretation of superqubits (or more generally superqudits) by introducing an auxiliary quantum mechanical system and combining it with the original Bose-Fermi system. We have enlarged the class of superqubits considered in [6], and promoted their Grassmann coefficients to Clifford dynamical variables, which are fermionic creation operators of the auxiliary quantum system. As in [6] the resulting superqubit has bosonic statistics, and quantum supergates are now unitary matrices with Clifford entries, rather than supergroup matrices. It would be worthwhile to investigate their LOCC, SLOCC and entanglement classes (cf. [6, 9] for Grassmann superqubits).

Supersymmetry can be implemented, the supersymmetry operator being one of the supergates. If the environment would act on supersymmetric squbits only via supersymmetric quantum gates (and this is a big if) then supersymmetry could provide a protection against decoherence.

## References

- [1] V. Coffman, J. Kundu and W. K. Wootters, *Distributed Entanglement*, Phys. Rev. A **61**, 052306 (2000) [arXiv:quant-ph/9907047].
- [2] A. Miyake and M. Wadati, *Multipartite entanglement and hyperdeterminants*, *Quant. Info. Comp.* **2** (Special) (2002) 540–555, arXiv:quant-ph/0212146.
- [3] A. Miyake, *Classification of multipartite entangled states by multi-dimensional determinants*, *Phys. Rev.* **A67** (2003) no. 1, 012108, arXiv:quant-ph/0206111.
- [4] A. Cayley, *On the theory of linear transformations*, Camb. Math. J., 4 193-209 (1845).
- [5] L. Castellani, P. A. Grassi and L. Sommovigo, *Triality Invariance in the N=2 Superstring*, Phys. Lett. B **678**, 308 (2009) [arXiv:0904.2512 [hep-th]].
- [6] L. Borsten, D. Dahanayake, M. J. Duff and W. Rubens, *Superqubits*, [arXiv:0908.0706 [quant-ph]].
- [7] O. Rudolph, *Super Hilbert Spaces*, Commun. Math. Phys. **214**, 449 (2000)
- [8] R. Haag, *Local quantum physics: Fields, particles, algebras, Berlin, Germany: Springer (1992) 356 p. (Texts and monographs in physics)*
- [9] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim and W. Rubens, *Black Holes, Qubits and Octonions*, Phys. Rept. **471**, 113 (2009) [arXiv:0809.4685 [hep-th]].